MOST SMALL p-GROUPS HAVE AN AUTOMORPHISM OF ORDER 2

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Abstract. Let $f(p,n)$ be the number of pairwise nonisomorphic p-groups of order p^n, and let $g(p,n)$ be the number of groups of order p^n whose automorphism group is a p-group. We prove that the limit, as p grows to infinity, of the ratio $g(p,n)/f(p,n)$ equals $1/3$ for $n = 6, 7$.

1. Introduction

In [8, p. 362], Mann poses the following question. If $f(p,n)$ is the number of pairwise nonisomorphic groups of order p^n and $g(p,n)$ the number of groups of order p^n whose automorphism group is a p-group, then does

$$\lim_{n\to\infty} \frac{g(p,n)}{f(p,n)} = 1?$$

Theorems of Helleloid-Martin and Martin suggest this ought to be true [6, 9].

Using the classifications of groups of order p^6 and p^7 developed by Newman, O'Brien, and Vaughan-Lee [10, 11], we have access to the prominent families (e.g. large isoclinism classes), allowing for asymptotic statements about these groups. We prove the following theorem.

Theorem 1.
$$\lim_{p\to\infty} \frac{g(p,6)}{f(p,6)} = \lim_{p\to\infty} \frac{g(p,7)}{f(p,7)} = \frac{1}{3}.$$

It is also sensible to test Mann's hypothesis on the current database of p-groups [10, 11]. This is done in [6, Table 1] for groups of order p^n where $p \leq 5$ and $n \leq 7$. There is a technical challenge when increasing the values of either p or n, and this has only recently become possible by work of Brooksbank, Wilson, and the author to improve isomorphism testing [3, 7, 14]. We expand known tables by including larger values of p and n, see Table 1. Even with the state of the art algorithms, our tables required several months of computation on a computer running Magma V2.21-5 with Intel Xeon W3565 3.20 GHz micro-processors.

2. Preliminaries

Throughout, all groups are finite. For $g, h, k \in G$, we set $[g,h] = g^{-1}h^{-1}gh$ and $[g,h,k] = [[g,h],k]$. Moreover, for $X, Y \subseteq G$, let $[X,Y] = \langle [x,y] \mid x \in X, y \in Y\rangle$. We set $\Omega(G) = \langle g \in G : g^p = 1\rangle$ and $G^p = \langle g^p : g \in G\rangle$. We let $\mathbb{Z}_p$ denote the cyclic group of order p.

Let $\gamma_1(G) = \eta_1(G) = G$ and for all $i \in \mathbb{Z}^+$ set $\gamma_{i+1}(G) = [\gamma_i(G), G]$ and $\eta_{i+1}(G) = [\eta_i(G), G]\eta_i(G)^p$. For a nilpotent group G, the *class* (p-*class*) of G

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p	p^6	p^7	p^8	p^9
2	211 (79.03%)	2,067 (88.79%)	54,463 (97.10%)	10,477,331 (99.84%)
3	30 (5.95%)	2,119 (22.76%)	1,002,216 (71.79%)	
5	65 (9.50%)	11,895 (34.68%)		
7	91 (10.58%)	42,208 (37.30%)		
11	189 (15.86%)	286,385 (38.15%)		
13	241 (16.33%)			
17	389 (20.01%)			
19	463 (20.45%)			

TABLE 1. The number of isomorphism types of p-groups whose automorphism group is a p-group.

is the largest index where $\gamma_i(G) \neq 1$ ($\eta_i(G) \neq 1$). If G is p-class c, then H is an *immediate descendant* of G if H is p-class $c+1$ and $G \cong H/\eta_{c+1}(H)$.

2.1. **Bilinear maps.** Let K be a field, and let U, V, and W be K-vector spaces. A K-bilinear map (K-bimap) is a function $\circ : U \times V \rightarrowtail W$ such that, for all $u, u' \in U$, $v, v' \in V$, and $k \in K$

$$(u + ku') \circ v = u \circ v + k(u' \circ v) \qquad \& \qquad u \circ (v + kv') = u \circ v + k(u \circ v').$$

The *radicals* of $\circ$ are $U^\perp = \{v \in V \mid U \circ v = 0\}$ and $V^\perp = \{u \in U \mid u \circ V = 0\}$. A bimap is *nondegenerate* when $U^\perp = V^\perp = 0$ and is *fully-nondegenerate* when, in addition to begin nondegenerate, $W = U \circ V$.

Two bimaps $\circ : V \times V \rightarrowtail W$ and $\bullet : V' \times V' \rightarrowtail W'$ are *pseudo-isometric* if there exists isomorphisms f and g making the diagram commute

$$\begin{array}{ccccc} \circ : V & \times & V & \longrightarrow & W \\ \downarrow f & & \downarrow f & & \downarrow g \\ \bullet : V' & \times & V' & \longrightarrow & W'. \end{array}$$

Additionally, bimaps $\circ$ and $\bullet$ are *isometric* if they are pseudo-isometric and if $W = W'$ and $g = 1$. The pseudo-isometry and isometry groups are denoted by $\Psi\mathrm{Isom}(\circ)$ and $\mathrm{Isom}(\circ)$ respectively.

Associated to bimaps is the adjoint ring

$$\mathrm{Adj}(\circ) = \{(f, g) \in \mathrm{End}(U) \times \mathrm{End}(V)^{\mathrm{op}} : \forall u \in U, v \in V, uf \circ v = u \circ gv\},$$

which plays a major role in computing $\Psi\mathrm{Isom}(\circ)$ and $\mathrm{Isom}(\circ)$ [3, 4].

2.2. **Isoclinism.** Groups G and H are *isoclinic* if there exists isomorphisms $\varphi : G/Z(G) \to H/Z(H)$ and $\hat{\varphi} : G' \to H'$ such that the following diagram commutes

$$\begin{array}{ccccc} [,]_G : G/Z(G) & \times & G/Z(G) & \longrightarrow & G' \\ \downarrow \varphi & & \downarrow \varphi & & \downarrow \hat{\varphi} \\ [,]_H : H/Z(H) & \times & H/Z(H) & \longrightarrow & H', \end{array}$$

see [5] for more details. When G is p-class 2, $G/Z(G)$ and G' are elementary abelian, and $[,]_G$ is a $\mathbb{Z}_p$-bilinear map. Hence, an isoclinism from G to H is a pseudo-isometry from the bimaps $[,]_G$ to $[,]_H$.

2.3. **Central extensions of elementary abelian p-groups.** Suppose G is a p-group of p-class 2, and let $\mathcal{I}$ be the isoclinism class containing G. Additionally, suppose $1 \to A \to G \to B \to 1$ is a central extension. Using the Universal Coefficients Theorem, we get a lower bound on the number of groups in $\mathcal{I}$; see [1] for a similar technique.

Theorem 2 (Universal Coefficients Theorem [13, Ch. 5])**.** *If A and B are abelian groups, then the following is a short exact sequence of groups*

$$1 \longrightarrow \mathrm{Ext}(B, A) \longrightarrow H^2(B, A) \longrightarrow \mathrm{Hom}(B \wedge B, A) \longrightarrow 1.$$

The groups $H^2(B, A)$ and $\mathrm{Ext}(B, A)$ can be interpreted as the set of all central and abelian extensions of A by B, respectively. Furthermore, $\mathrm{Ext}(B, A)$ is isomorphic to $\mathrm{Hom}(B, A)$.

We are concerned with the case when $A = \gamma_2(G) = Z(G) \cong \mathbb{Z}_p^a$ and $B = G/\gamma_2(G) \cong \mathbb{Z}_p^b$. For groups $G, H \in \mathcal{I}$, we identify $G/\gamma_2(G) = B = H/\gamma_2(H)$ and $\gamma_2(G) = A = \gamma_2(H)$, so that an isomorphism from, say, $G/\gamma_2(G)$ to $H/\gamma_2(H)$ is contained in $\mathrm{Aut}(B) \cong \mathrm{GL}(b, p)$. From $h : B \wedge B \to A$, we construct a fully-nondegenerate, alternating $\mathbb{Z}_p$-bimap $[,] = \wedge^h : B \times B \rightarrowtail A$ such that $[b, b'] = (b \wedge b')h$. The group $\Psi\mathrm{Isom}([,])$ acts on $\mathrm{Hom}(B, A)$: for $(\varphi, \hat{\varphi}) \in \Psi\mathrm{Isom}([,])$ and $f \in \mathrm{Hom}(B, A)$,

$$f^{(\varphi,\hat{\varphi})} = \varphi^{-1} f \hat{\varphi}.$$

Suppose G and H are central extensions of A by B, determined by $f, f' \in \mathrm{Hom}(B, A)$ and $h, h' \in \mathrm{Hom}(B \wedge B, A)$ respectively. If there exists $(\varphi, \hat{\varphi}) \in \mathrm{GL}(b, p) \times \mathrm{GL}(a, p)$ such that $(\varphi, \hat{\varphi})$ is a pseudo-isometry from $\wedge^h$ to $\wedge^{h'}$ and $f' = \varphi^{-1} f \hat{\varphi}$, then $G \cong H$. This implies that $|\mathcal{I}| \geq |\mathrm{Hom}(B, A)|/|\Psi\mathrm{Isom}([,])|$.

We now consider how this relates to the automorphism group of G. We let $C_{\mathrm{Aut}(G)}(G/\gamma_2(G))$ denote the subgroup of $\mathrm{Aut}(G)$ that induces the identity on the quotient $G/\gamma_2(G)$, and hence, is a p-group. The following is an exact sequence

$$1 \longrightarrow C_{\mathrm{Aut}(G)}(G/\gamma_2(G)) \longrightarrow \mathrm{Aut}(G) \longrightarrow \Psi\mathrm{Isom}([,]).$$

Since G is a central extension of A by B, there exists $f \in \mathrm{Hom}(B, A)$ and $h \in \mathrm{Hom}(B \wedge B, A)$ for G. Therefore, the image of $\mathrm{Aut}(G)$ in $\Psi\mathrm{Isom}([,])$ is the subgroup stabilizing f. Since G is class 2, the following is an exact sequence

$$1 \longrightarrow C_{\mathrm{Aut}(G)}(G/\gamma_2(G)) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Stab}_{\Psi\mathrm{Isom}([,])}(f) \longrightarrow 1. \tag{3}$$

3. Lower bounds

We prove that the limit in Theorem 1 is bounded below by 1/3 in two different cases: $n = 7$ and $n = 6$.

3.1. **The lower bound for $n = 7$.** We first consider the groups of order p^7. For a fixed odd prime p, define

$$P = \langle a, b, c, d \mid [c, a][d, b]^{-1}, [d, a], [c, b], \text{class } 2, \text{exponent } p\rangle. \tag{4}$$

Note that $Z(P) = \gamma_2(P) = \Phi(P)$. We consider the set of groups isoclinic to P, denoted $\mathcal{I}$. If $G \in \mathcal{I}$, then $\Psi\mathrm{Isom}([,]_G) \cong \Psi\mathrm{Isom}([,]_P)$, so set $[,] = [,]_P$.

Let $V = P/P'$ and $W = P'$. The adjoint algebra of $\circ : V \times V \rightarrowtail W$ is a $*$-algebra with the symplectic involution:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \overline{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}.$$

More specifically,

$$A = \mathrm{Adj}(\circ) = \left\{ \left(\begin{bmatrix} M & 0 \\ 0 & \overline{M}^t \end{bmatrix}, \begin{bmatrix} \overline{M} & 0 \\ 0 & M^t \end{bmatrix} \right) : M \in \mathbb{M}_2(\mathbb{Z}_p) \right\}. \tag{5}$$

Observe that

$$V \wedge_A V := V \otimes V / \langle (uf) \otimes v - u \otimes (vg) : \forall u, v \in V, \forall (f,g) \in A \rangle \cong W. \tag{6}$$

If φ is such an isomorphism then $(1, \varphi)$ is a pseudo-isometry from $\wedge_A$ to $\circ$. Therefore, $\Psi\mathrm{Isom}(\wedge_A) \cong \Psi\mathrm{Isom}(\circ)$. Furthermore, $\mathrm{Adj}(\circ)$ is a $*$-simple algebra isomorphic to $\mathbb{M}_2(\mathbb{Z}_p)$ with the symplectic involution and, by equation (5), is irreducibly represented on a two dimensional vector space. By [4, Theorems 1.5 & 4.5]

$$\Psi\mathrm{Isom}(\circ) \cong \mathrm{GSp}(2,p) \otimes_{\mathbb{Z}_p} \mathrm{GL}(2,p), \tag{7}$$

where $\mathrm{GSp}(n,p) = \mathrm{Sp}(n,p) \rtimes \mathbb{Z}_p^\times$.

The number of pairwise non-isomorphic groups in the isoclinism class $\mathcal{I}$ is bounded below by $|\mathrm{Hom}(\mathbb{Z}_p^4, \mathbb{Z}_p^3)|/|\Psi\mathrm{Isom}([,]_P)|$. Hence, by equation (7), there are at least

$$\frac{|\mathrm{Hom}(\mathbb{Z}_p^4, \mathbb{Z}_p^3)|}{|\mathrm{GSp}(2,p) \otimes_{\mathbb{Z}_p} \mathrm{GL}(2,p)|} = p^5 + p^4 + 3p^3 + 3p^2 + 6p + 6 + o(1) \tag{8}$$

groups. We state a lemma which follows from the orbit-stabilizer theorem.

Lemma 9. *Let $\mathcal{I}$ be an isoclinism class of groups which are extensions of elementary abelian groups A by B, and let $\Psi\mathrm{Isom}(\mathcal{I})$ be the pseudo-isometry group. If s is the number of orbits in $\mathcal{I}$ such that* $\mathrm{Stab}_{\Psi\mathrm{Isom}(\mathcal{I})}(\mathrm{Hom}(B,A)) = 1$, *then*

$$s \geq \frac{|\mathrm{Hom}(B,A)|}{|\Psi\mathrm{Isom}(\mathcal{I})|}.$$

Proposition 10.

$$\lim_{p \to \infty} \frac{g(p,7)}{f(p,7)} \geq \frac{1}{3}.$$

Proof. Let P be defined as in equation (4). By equation (8) and Lemma 9, the number of homomorphisms in $\mathrm{Hom}(\mathbb{Z}_p^4, \mathbb{Z}_p^3)$ with a trivial stabilizer is bounded below by $p^5 + O(p^4)$. From the short exact sequence in (3), $g(p,7) \geq p^5 + O(p^4)$. From [11, Theorem 1], if $p > 5$, then

$$f(p,7) = 3p^5 + O(p^4).$$

Therefore, for $p > 5$, $g(p,7)/f(p,7) \geq 1/3 - \epsilon_p$, where $\epsilon_p \to 0$ as $p \to \infty$. □

3.2. **The lower bound for p^6.** Consider the set of groups with the following presentations. For $0 \leq r \leq s < p$, let

$$\begin{aligned} P(r,s) = \langle a,b,c \mid\ & [c,a][b,a,a]^{-1}, [c,b], [b,a,c], \\ & a^p[b,a,b]^{-r}, b^p[b,a,b]^{-s}, c^p[b,a,a]^{-1}[b,a,b]^{-1}, p\text{-class } 3 \rangle. \end{aligned}$$

Additionally, for $0 \leq t \leq (p-1)/2$, $0 \leq u < p$, and a non-square $\omega \in \mathbb{Z}_p$, let

$$\begin{aligned} Q(t,u) = \langle a,b,c \mid\ & [c,a][b,a,b]^{-1}, [c,b][b,a,a]^{-\omega}, [b,a,c], \\ & a^p[b,a,a]^{-t}, b^p[b,a,a]^{-u}, c^p[b,a,b]^{-1}, p\text{-class } 3 \rangle. \end{aligned}$$

We show that most of the groups $P(r,s)$ and $Q(t,u)$ have an automorphism group which is a p-group. However, there are a few cases with p'-automorphisms.

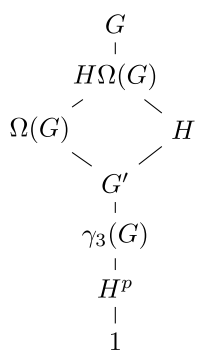
Lemma 11. *The groups $P(r,r)$ and $Q(0,u)$ have a p'-automorphism.*

Proof. The map of $P(r,r)$ which sends a to $b^{-1}c^r$, b to $a^{-1}c^r$, and c to $c[b,a]^{-1}$ induces an automorphism. In addition, the map of $Q(0,u)$ which fixes a and c but inverts b induces an automorphism. □

To show that nearly all groups $P(r,s)$ and $Q(t,u)$ have an automorphism group a p-group, we compute the automorphism groups of their Lie rings using the Lazard-Mal'cev correspondence. For details on this technique, see [10, Section 4]. To make the computations easier, we determine a characteristic composition series for the groups. Therefore, we show that the corresponding parabolic subgroup $A(G) \leq \mathrm{GL}(3,p)$ must be trivial. Because the torus is not split for these groups, this is not a trivial computation. In the proceeding lemma, we solve nonlinear equations to compute the automorphism group of the associated Lie rings.

Lemma 12. *If $G \in \{P(r,s), Q(t,u) \mid r \neq s, t \neq 0\}$, then* $\mathrm{Aut}(G)|_{G/\Phi(G)} = 1$.

Proof. Note that $H = \langle c, G' \rangle$ is characteristic as it is the radical of $[,]_G : G/G' \times G/G' \rightarrowtail G'/\gamma_3(G)$. A characteristic composition series of G is given in Figure 1.



For $G = P(r,s)$:

$$H^p = \langle [b,a,a][b,a,b] \rangle,$$
$$\Omega(G) = \langle ab^{-r/s}, G' \rangle.$$

For $G = Q(t,u)$:

$$H^p = \langle [b,a,b] \rangle,$$
$$\Omega(G) = \langle a^{-u/t}b, G' \rangle.$$

Figure 1. Some characteristic subgroups of G.

First, consider $G = P(r,s)$, where $r \neq s$. We use

$$\left(b, ab^{-r/s}, c, [b,a]^{-1}[b,a,b]^{r/s}, [b,a,a], [b,a,a][b,a,b]\right)$$

as a consistent generating set, determined by the composition series in Figure 1. Therefore, the Lie ring of $P(r,s)$ has the following presentation

$$\begin{aligned} R_P = \langle x_1, \ldots, x_6 \mid\ & px_1 = s(-x_5 + x_6),\ px_3 = x_6, \\ & x_2x_1 = x_4,\ x_3x_2 = x_5,\ x_4x_1 = x_5 - x_6, \\ & x_4x_2 = -(r/s+1)x_5 + (r/s)x_6 \rangle, \end{aligned}$$

all omitted power-commutator relations are assumed to be trivial.

Suppose that $\alpha, \ldots, \zeta \in \mathbb{Z}_p$ and the following induces an automorphism of R_P

$$\begin{aligned} x_1 &\mapsto \alpha x_1 + \beta x_2 + \gamma x_3, & x_4 &\mapsto \alpha\delta x_4 - \gamma\delta x_5, \\ x_2 &\mapsto \delta x_2, & x_5 &\mapsto (\delta\epsilon - \delta\zeta(r/s+1))x_5 + \delta\zeta r/s x_6, \\ x_3 &\mapsto \epsilon x_3 + \zeta x_4, & x_6 &\mapsto \epsilon x_6. \end{aligned}$$

Such an automorphism must satisfy the following equations

$$\begin{aligned}
\alpha\zeta + \beta\epsilon &= \beta\zeta(r/s+1), & \alpha^2\delta - \alpha\beta\delta r/s &= \epsilon - \delta\zeta r/s,\\
\beta\zeta r/s &= \alpha\zeta, & \epsilon r/s - \alpha\delta^2 r/s &= \delta\zeta r/s(r/s+1),\\
\delta\epsilon - \alpha &= \delta\zeta(r/s+1), & \alpha\delta^2(r/s+1) &= \delta\epsilon(r/s+1) - \delta\zeta(r/s+1)^2,\\
\gamma - \alpha s &= s\epsilon - \delta\zeta r, & \alpha\beta\delta(r/s+1) &= \alpha^2\delta + \delta\zeta(r/s+1) - \delta\epsilon.
\end{aligned}$$

We have two cases, either $\zeta = 0$ or $\zeta \neq 0$. If $\zeta = 0$, then only the trivial automorphism satisfies the above equations. If $\zeta \neq 0$, then

$$(r/s-1)(r/s+1)^2 = 0 \qquad r/s = 1.$$

Therefore, since $r \neq s$, $\mathrm{Aut}(G)|_{G/\Phi(G)} = 1$.

Finally, consider $G = Q(t,u)$, where $t \neq 0$, and use

$$\left(a, a^{-u/t}b, c, [b,a], [b,a,a], [b,a,b]\right)$$

as a consistent generating set. The Lie ring for G has the following presentation

$$\begin{aligned}
R_Q = \langle x_1, \dots, x_6 \mid\ & px_1 = tx_5,\ px_3 = x_6,\\
& x_2x_1 = x_4,\ x_3x_1 = x_6,\ x_3x_2 = \omega x_5 - (u/t)x_6,\\
& x_4x_1 = x_5,\ x_4x_2 = -(u/t)x_5 + x_6\rangle,
\end{aligned}$$

again, all trivial power-commutator relations are omitted.

Suppose $\alpha, \dots, \zeta \in \mathbb{Z}_p$ and the following induces an automorphism of R_Q

$$\begin{aligned}
x_1 &\mapsto \alpha x_1 + \beta x_2 + \gamma x_3, & x_4 &\mapsto \alpha\delta x_4 + \omega\gamma\delta x_5 - \gamma\delta(u/t)x_6,\\
x_2 &\mapsto \delta x_2, & x_5 &\mapsto \alpha t x_5 + \gamma x_6,\\
x_3 &\mapsto \epsilon x_3 + \zeta x_4, & x_6 &\mapsto \epsilon x_6.
\end{aligned}$$

Such an automorphism must satisfy the following equations

$$\begin{aligned}
\omega\beta\epsilon + \alpha\zeta &= \beta\zeta(u/t), & \alpha^2\delta &= \alpha\beta\delta(u/t) + \alpha t,\\
\alpha\epsilon + \beta\zeta &= \epsilon + \epsilon\beta(u/t), & \gamma &= \alpha\beta\delta,\\
\omega\delta\epsilon &= \omega\alpha t + \delta\zeta(u/t), & \alpha u &= \alpha\delta^2(u/t),\\
\delta\zeta + \epsilon(u/t) &= \omega\gamma + \delta\epsilon(u/t), & \epsilon &= \alpha\delta^2 + \gamma(u/t).
\end{aligned}$$

If $\beta = 0$ then only the trivial automorphism satisfies the above equations. If $\beta \neq 0$, then the above equations cannot be satisfied as $t \neq 0$. Therefore, when $t \neq 0$, we must have $\beta = 0$, and hence, $\mathrm{Aut}(G)|_{G/\Phi(G)} = 1$. □

Proposition 13.

$$\lim_{p\to\infty} \frac{g(p,6)}{f(p,6)} \geq \frac{1}{3}.$$

Proof. There are p^2 groups isomorphic to $P(r,s)$ or $Q(t,u)$ for the various parameter values [12, p.18, (6.172) & (6.179)]. From Lemma 11, only $2p$ of those groups have an involution, and from Lemma 12, the remaining automorphism groups are p-groups because $\mathrm{Aut}(G)|_{G/\Phi(G)} = 1$. Therefore, $g(p,6) \geq p^2 - 2p$. By [10, Theorem 1], if $p \geq 5$, then

$$f(p,6) = 3p^2 + O(p),$$

so $g(p,6)/f(p,6) \geq 1/3 - \epsilon_p$, where $\epsilon_p \to 0$ as $p \to \infty$. □

4. Upper bounds

We prove that the limit of the ratio in Theorem 1 is bounded above by 1/3, so together with Propositions 10 & 13, this finishes the proof of Theorem 1. The proof of the following proposition significantly depends on finding large isoclinism classes in the classifications of groups of order p^6 and p^7 [10,11]. Throughout the proof we reference the Lie rings in the database of Eick and Vaughan-Lee [12]; however, we present the group version of the Lie rings, using the Lazard correspondence.

Proposition 14. *For $n \in \{6, 7\}$,*

$$\lim_{p\to\infty} \frac{g(p,n)}{f(p,n)} \leq \frac{1}{3}.$$

Proof. Fix $p > 5$. First we consider groups of order p^6 that are immediate descendants of

$$P = \langle a, b, c \mid [c, a] = [c, b] = 1, \text{class } 2, \text{exponent } p\rangle.$$

There are $(p-1)(p-3)$ immediate descendants of P, of order p^6, that satisfy equivalent relations to the following [12, p. 17, (6.163) & (6.164)]

$$\begin{aligned} [c, a] &= [b, a, a], \\ [c, b] &= [b, a, c] = c^p = 1. \end{aligned} \tag{15}$$

Moreover, there are $p^2 + (p + 1)/2 - \gcd(p - 1, 4)/2$ immediate descendants of P, of order p^6, that satisfy equivalent relations to the following [12, p. 18, (6.178)].

$$\begin{aligned} [c, a] &= [b, a, b], \\ [c, b] &= [b, a, a]^\omega, \\ c^p &= 1. \end{aligned} \tag{16}$$

We assume that ω is a non-square modulo p. Although the above groups satisfy more relations, the map that inverts a and b and fixes c induces an automorphism of each of these groups.

Now, we consider groups of order p^7 that are immediate descendants of

$$Q = \langle a, b, c \mid [c, b] = 1, \text{class } 2, \text{exponent } p\rangle.$$

There are $p^5 + p^4 + p^3 + O(p^2)$ immediate descendants of Q, of order p^7, that satisfy equivalent relations to the following [12, pp. 89–90, (7.773) & (7.774)]

$$\begin{aligned} 1 = [c, b] &= [b, a, c], \\ [b, a, a] &= [c, a, c], \\ [c, a, a] &= [b, a, b]^\omega. \end{aligned} \tag{17}$$

When $p = 2 \mod 3$, we set $\omega = 1$; otherwise, ω is either a non-square or $\omega = 1$. In addition, there are $p^5 + p^4 + p^3 + O(p^2)$ immediate descendants of Q, in total, that satisfy equivalent relations to one of the following [12, p. 92, (7.779) & (7.780)].

$$\begin{aligned} 1 = [c, b] &= [b, a, a], \\ [c, a, a] &= [b, a, b]^x [b, a, c], \\ [c, a, c] &= [b, a, b]^\omega, \end{aligned} \tag{18}$$

If $p = 1 \bmod 3$, set $x = 0$; otherwise, the values of x are found in [12, p. 92]. Again, these groups satisfy more relations, but the map that inverts the generators a, b, and c induces an automorphism of these groups.

From [10, Theorem 1] and [11, Theorem 1], if $p > 5$, then

$$f(p,6) = 3p^2 + O(p),$$
$$f(p,7) = 3p^5 + O(p^4).$$

We have shown that $g(p,6) \leq f(p,6) - 2p^2 - O(p)$ and $g(p,7) \leq f(p,7) - 2p^5 - O(p^4)$. Therefore, if $n \in \{6,7\}$, then $g(p,n)/f(p,n) \leq 1/3 + \epsilon_p$, where $\epsilon_p \to 0$ as $p \to \infty$. □

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